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DISCUSSION

Comments on: Subsampling weakly dependent time series and application to extremes

S.N. Lahiri · S. Mukhopadhyay

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1 Introduction

Professors Doukhan, Prohl, and Robert are to be congratulated for their work on extending the validity of the subsampling method to a much wider class of processes compared to the existing literature that typically requires the processes to be strongly mixing (cf. Politis et al. 1999). As described in Sects. 1 and 2, many common time series models, including the ARMA models, often fail to satisfy the strong mixing condition but they typically satisfy the η- and λ-weak dependence conditions of Doukhan and Louhichi (1999) considered in this paper. As a result, extending the validity of the subsampling method under suitable η- and λ-weak dependence conditions is an important contribution. Expectedly, the smooth version of the subsampling estimator is especially suited to the form of the η- and λ-weak dependence conditions which give covariance bounds for smooth functions of the observations. This is one reason why the validity of the smooth subsampling estimator in Theorem 1 holds under *weaker* conditions than those for the rough subsampling estimator in Theorem 3. However, from the applications point of view, it is worth noting that while smoothing is known to play an important role in the resampling methodology in certain inference problems (e.g., inference on quantiles), caution must be exercised in cases where the limit distribution of the (unbootstrapped) statistic has points of discontinuity.

The authors also prove validity of the (smooth) subsampling estimator for the sample maximum under different sets of η- and λ-weak dependence conditions, for both the overlapping and the non-overlapping cases. This is an important problem where its natural competitor, namely, the block bootstrap method (cf. Künsch 1989

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S.N. Lahiri (✉) · S. Mukhopadhyay
Texas A & M University, College Station, USA
e-mail: snlahiri@stat.tamu.edu

and Liu and Singh 1992) does not always provide the correct answer. More precisely, it is well known (cf. Athreya et al. 1999; Lahiri 2003) that even under stronger strong mixing conditions, the block bootstrap methods do not provide a valid approximation to the distribution of the sample maximum when the resample size equals the sample size. Like in many other similar problems (cf. Bickel et al. 1997), a consistent block bootstrap approximation can be generated by choosing a resample size m that grows at a slower rate than the sample size n, i.e., when

$$m = o(n),$$

which is referred to as the 'm out of n' block bootstrap. In the rest of this note, we compare performance of the subsampling and the 'm out of n' block bootstrap methods for the sample maximum under the η-mixing condition.

2 Theoretical properties

For completeness, we briefly describe the 'm out of n' (overlapping or moving) block bootstrap (MBB) method using the notation of the main paper. Given observations $X_1, \dots, X_n$ from a stationary time series, let $\{Y_{b,i} : i = 0, \dots, N\}$ denote the overlapping blocks of size b, as defined in (3), where $N = n - b$. The 'm out of n' MBB resamples $k \geq 1$ blocks with replacement from this collection to generate a bootstrap sample of size $m = bk$, which we shall denote as $X_1^*, \dots, X_m^*$. Then, the 'm out of n' MBB estimator of the distribution of a statistic $R_n = r_n(X_1, \dots, X_n)$ is given by the conditional distribution of $R_{m,n}^* \equiv r_m(X_1^*, \dots, X_m^*)$, given the X_i's. In particular, the 'm out of n' MBB estimator of $\mathbb{H}_n(x) \equiv P([M_n - v_n]/u_n \leq x)$ is given by

$$\widehat{\mathbb{H}}_{m,n}(x) \equiv P_*\big([M_{m,n}^* - \tilde{v}_m]/\tilde{u}_m \leq x\big),$$

where $M_{m,n}^* = \max\{X_1^*, \dots, X_m^*\}$ is the bootstrap version of $M_n = \max\{X_1, \dots, X_n\}$, P_* denotes the conditional probability given the X_i's, and where $\tilde{v}_m$ and $\tilde{u}_m$ are analogs of the centering and scaling constants v_n and u_n, respectively. A possible choice of $\tilde{v}_m$ and $\tilde{u}_m$ that leads to a valid approximation is given by $\tilde{v}_m = v_m$ and $\tilde{u}_m = u_m$, which normalizes $M_{m,n}^*$ at the level of the subsamples (that are of size m). A more standard choice, especially when these constants are unknown, are given by replacing F and n in the definitions of v_n and u_n (given right after (10)) with the empirical distribution function F_n (say) of $X_1, \dots, X_n$ and with m, respectively. Finally, when $k = 1$, the 'm out of n' MBB reduces to the *rough* subsampling estimator of $\mathbb{H}_n(\cdot)$. In this case, the standard choice of $\tilde{v}_m$ is given by $\tilde{v}_m = F_n^{\leftarrow}(1 - n^{-1})$ while $\tilde{u}_m$ is chosen as in the last case. For all these variants, the key requirement for the 'm out of n' MBB to work is that

$$u_m^{-1}\big[|\tilde{v}_m - v_m| + |\tilde{u}_m - u_m|\big] \to_p 0 \quad \text{as } n \to \infty, \tag{1}$$

which we shall assume for the rest of this discussion. We, however, point out that for the data-based choices of $\tilde{v}_m$ and $\tilde{u}_m$ mentioned above, (1) can be proved using the arguments developed in Fukuchi (1994). Then, we have the following result:

Theorem *Suppose that* (1) *holds and that the conditions of Theorem* 5 *hold for the η-weakly dependent case. Also suppose that $b = b_n$, $k = k_n$, and $m = m_n \equiv b_n k_n$ are* *such that* (i) $b_n^{-1} + n^{-1}m_n = o(1)$; (ii) $\lim_{p\to\infty} \limsup_{n\to\infty} k[mu_m\eta(pb)]^{1/2} = 0$; (iii) $kP(M_b > w_m(x)) = O(1)$ *for each* $x \in \mathbb{R}$. *Then*,

$$\sup_{x\in\mathbb{R}} \left|\hat{\mathbb{H}}_{m,n}(x) - \mathbb{H}_n(x)\right| \to_p 0 \quad \textit{as } n \to \infty.$$

A sketch of the proof of the theorem is given in the Appendix. To briefly comment on the conditions, (i) is a very standard condition on the block size b in the block bootstrap literature, (ii) is similar to the η-weak dependence condition in Theorem 5, and (iii) is implied by $P(M_b > w_m(x)) \le b\bar{F}(w_m(x))$ for each $x \in \mathbb{R}$, which is akin to (11). Note that the theorem requires the block size b to grow to infinity with n and the resample size m to grow at a rate slower than n, but it does not otherwise impose any conditions on the number of resamples k. In particular, the consistency holds for the case $k = 1$, i.e., for the rough subsampling estimator and, more generally, for the 'm out of n' MBB under the conditions of the Theorem. This extends the results of Athreya et al. (1999) to the η-dependence case. Similar result also holds for the nonoverlapping version of the 'm out of n' block bootstrap, which we omit to save space.

3 Numerical results

We now report the results from a simulation study comparing the performance of the 'm out of n' MBB and the (rough) subsampling methods for the sample maximum. As in the main paper, we considered the model given by (1) and drew 1000 Monte Carlo samples with sample sizes $n = 40$, 200, and 2000. For each sample $(X_1, X_2, \ldots, X_n)$, we resampled the data randomly 1000 times for constructing the block bootstrap estimator. The block lengths we have considered here are (closest integers to) $b_1 = 2n^{1/3}$ and $b_2 = \sqrt{(1.25)n}$; The second block length b_2 equals 50 for $n = 2000$, matching the choice of b in the main paper. The following table reports the (scaled) global error (using a version of the Cramer–von Mises distance) of approximating the true distribution $H_n(\cdot)$ by the 'm out of n' MBB and the subsampling methods.

Results in Table 1 show that the block bootstrap produces more accurate global approximation to the true distribution than the subsampling method for all combinations of n and b. Further, for both methods, the overlapping versions have slightly

Table 1 Global errors (in %) for approximating the true distribution $\mathbb{H}_n$ by the 'm out of n' MBB and the subsampling methods (denoted as MBB and SS, respectively, in the table)

		Overlapping			Non-overlapping		
		$n = 50$	$n = 200$	$n = 2000$	$n = 50$	$n = 200$	$n = 2000$
MBB	b_1	1.03	0.84	0.37	0.9	0.78	0.36
MBB	b_2	0.3	0.2	0.15	0.22	0.18	0.14
SS	b_1	4.66	2.89	0.9	4.86	2.7	0.88
SS	b_2	3.4	1.55	0.4	3.42	1.48	0.42

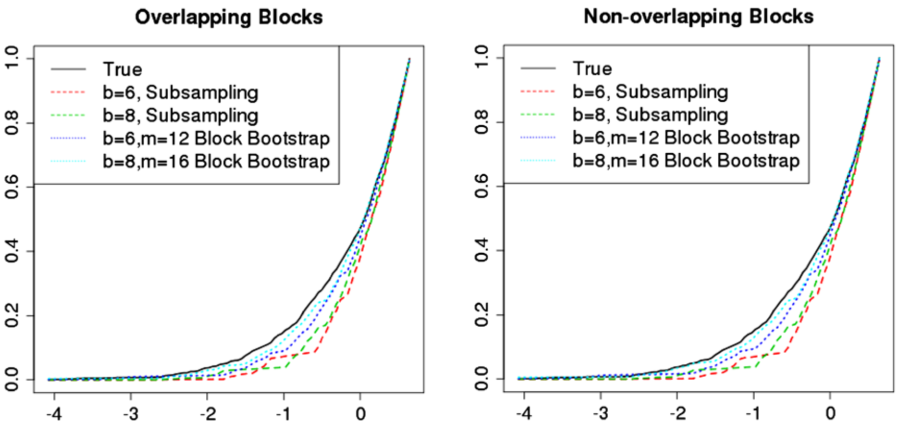

Fig. 1 Block bootstrap and subsampling estimators of the true distribution $\mathbb{H}_n$ for $n = 50$

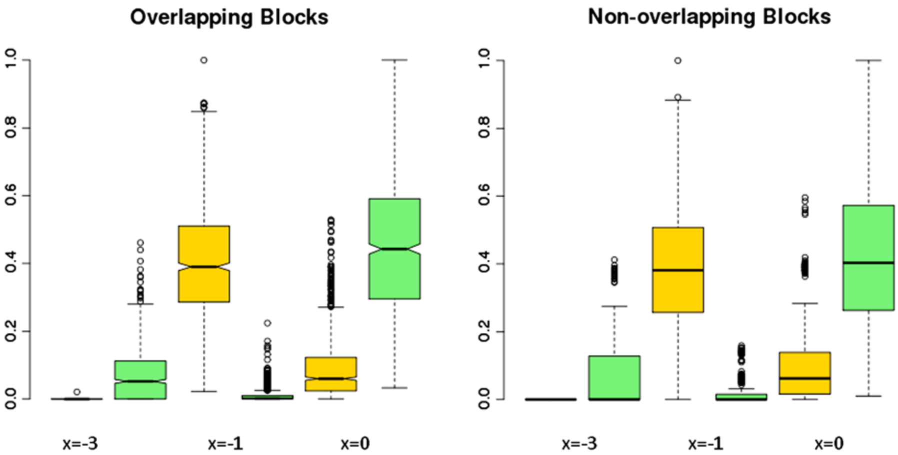

Fig. 2 Box-plots of the block bootstrap (even numbered) and subsampling estimators (odd numbered) of $\mathbb{H}_n(x)$ at $x = -3, -1, 0$ for $n = 50$

better performance than the non-overlapping versions. This is also evident from Fig. 1 which gives the mean CDF curves for the two methods for $n = 50$, based on 1000 simulation runs. We also considered local performance of the two methods, comparing the estimators of the CDF $\mathbb{H}_n(x)$ at different values of x. Figure 2 gives the box-plots of the estimators based on 1000 simulation runs for $n = 50$.

From Fig. 2, it follows that the estimators based on the non-overlapping versions of each method have higher variability than the corresponding overlapping versions, like in the case of smooth functions of means (cf. Lahiri 1999). Further, the variabilities of the subsampling estimators are much smaller than their block bootstrap counterparts at the tails ($x = -3, 0$), but the pattern reverses in the middle ($x = -1$). We observed a similar behavior also for $n = 200$ and $n = 2000$ (not shown here).

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Appendix

Here we provide an outline of the proof of the Theorem. First, note that by the continuity of the limit law $\mathbb{H}(\cdot)$ and by a subsequence argument, it is enough to show that for each fixed x,

$$\hat{\mathbb{H}}_{m,n}(x) - \mathbb{H}(x) \to_p 0.$$

Fix $x \in \mathbb{R}$. Write $M_{i,b} = \max Y_{i,b}$ and $M_b^* = \max\{X_1^*, \ldots, X_b^*\}$, the maximum over a single resampled block. Also, let $\mathbb{H}_b^\dagger(x) = P_*([M_b^* - v_m]/u_m \le x)$. Then, by (1) and the independence of the resampled blocks, it is easy to check that

$$\hat{\mathbb{H}}_{m,n}(x) = \left[\mathbb{H}_b^\dagger(x)\right]^k + o_p(1) = \left(1 + \frac{[k(1 - \mathbb{H}_b^\dagger(x)]}{k}\right)^k + o_p(1),$$

which, by Theorem 5 and Condition (i), converges to $\mathbb{H}(x)$ in probability provided $k^2 \operatorname{Var}(\mathbb{H}_b^\dagger(x)) \to 0$. Note that, by stationarity, with $w_m(x) = u_m^{-1}x + v_m$,

$$\begin{aligned}
&k^2 \operatorname{Var}\bigl(\mathbb{H}_b^\dagger(x)\bigr) \\
&\quad \le C_1 N^{-1} k^2 \sum_{j=0}^{N} \bigl|\operatorname{Cov}\bigl(\boldsymbol{I}\bigl(M_{0,b} > w_m(x)\bigr), \boldsymbol{I}\bigl(M_{j,b} > w_m(x)\bigr)\bigr)\bigr| \\
&\quad \le C_2 N^{-1} k^2 \Biggl[pbP\bigl(M_{0,b} > w_m(x)\bigr) \\
&\qquad + \sum_{j=(p+1)b}^{N} \bigl|\operatorname{Cov}\bigl(\boldsymbol{I}\bigl(M_{0,b} > w_m(x)\bigr), \boldsymbol{I}\bigl(M_{j,b} > w_m(x)\bigr)\bigr)\bigr| \Biggr] \\
&\quad \equiv I_{1n}(p) + I_{2n}(p) \quad \text{(say)},
\end{aligned}$$

where $C_1, C_2, \ldots \in (0, \infty)$ are constants. By Conditions (i) and (iii), it follows that $I_{1n}(p) = O(m/n) = o(1)$ for every fixed $p \ge 1$. And, by retracing the arguments in the proof of Theorem 5, one can show that for any $j \ge pb$, $p > 1$, and $\alpha_{jn} > 0$,

$$\begin{aligned}
&\bigl|\operatorname{Cov}\bigl(\boldsymbol{I}\bigl(M_0 > w_m(x)\bigr), \boldsymbol{I}\bigl(M_j > w_m(x)\bigr)\bigr)\bigr| \\
&\quad \le 2b\bigl(u_m \alpha_{jn}^{-1} \eta(j - b) + 2\alpha_{jn}\bigl[(1 + \gamma x)_+^{-\frac{1}{\gamma}-1}/m\bigr]\bigr).
\end{aligned}$$

Setting $\alpha_{jn} = [m u_m \eta(j - b)]^{1/2}$ and noting that $m = kb$, we have

$$I_{2n}(p) \le C_3 N^{-1} k^2 \sum_{j=[p+1]b}^{N} \bigl[b m^{-1/2} u_m^{1/2} \eta(j - b)^{1/2}\bigr] \le C_3 k m^{1/2} u_m^{1/2} \eta(pb)^{1/2},$$

which, by Condition (iii), goes to zero by first letting $n \to \infty$ and then $p \to \infty$. This completes the proof of the theorem.

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